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THE DETERMINATION OF THE MASS OF A PLANET FROM THE RELATIVE POSITION OF TWO SATELLITES.

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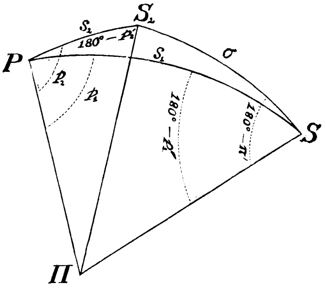
ONE of the easiest methods of determining the mass of a planet is from the observations of its satellites. The two principal elements that are needed are the periodic time of the satellite about its primary, and the semi-major axis of its orbit. Thus, if M and m are the masses of the sun and the planet, and τ_0 and τ are the periodic times of the planet and the satellite, we shall have very nearly,

$$\frac{m}{M} = \frac{\tau_0^2}{\tau^2} \cdot \frac{a^3}{a_0^3},$$

a_0 and a being the mean distances of the planet from the sun and of the satellite from its primary. The above is the simplest form of this equation, and should it be necessary in any case to consider the elliptical figure of the planet, and its mass when added to that of the sun, these small corrections can be applied easily. Of the two principal elements the periodic time of a satellite can be found with sufficient accuracy by comparing observations of it made at distant epochs, and hence the error arising from this source may be considered as vanishing. But with respect to the semi-major axis the case is different. For the determination of this element we have to depend on micrometrical measurements, and several sources of error present themselves. Thus the value in arc of one turn of the screw of the micrometer must be known very exactly, since this value will frequently be multiplied by a large factor, and of course an error in the value of the screw will also be multiplied by the same factor. With care on the part of the astronomer this error may be reduced to a very small quantity; and yet it is surprising how often such errors remain in micrometers that have been a long time in use. But there is another error in micrometrical measurements that is not easily got rid of, and this is the personal error of the observer. This error

will sometimes amount to half a second of arc, and since it is not eliminated by the repetition of observations, each observer will have his own value of the mean distance of a satellite, and hence his own mass of a planet. This personal error may be still greater, if, as is generally the case, we have to do with the disk of a planet. In this case we must bisect the disk with one of the wires, or else measure from the limb of the planet, and both operations are difficult and tend to increase the constant error of the measurement.

On account of this difficulty Struve, the present Director of the Pulkowa Observatory, has proposed that observations be made of the relative positions of two satellites for the purpose of correcting their elements, and in this way deducing the mass of the planet. Mr. Backlund, one of the astronomers of the Pulkowa Observatory, has already begun a series of measurements of the relative positions of the satellites of Jupiter with a heliometer, and it is said that these observations are two or three times as accurate as those made by reference to the limb of the planet, and moreover that the complete observation of the relative position of two satellites requires only a third part of the time that is necessary to determine the position of one of them with respect to the center of the planet. Mr. Backlund has published an outline of this method, which is theoretically very simple. Let P be the place of the planet projected on the heavens, Π the pole of the heavens, S_1 and S_2 the satellites, and denote the angles and sides of the spherical triangle, as in the figure.



The angles of position and distances are designated generally by p and s, with proper subscripts, and the relative angle of position and the distance of the two satellites by Π and σ. Then

$$\text{angle } S_1PS_2 = p_2 - p_1, \qquad \text{angle } S_2S_1P = p_1' - \pi',$$

and the spherical triangle S_1PS_2 gives the equations,

$$\sin\sigma \sin(p_1' - \pi') = \sin s_2 \sin(p_2 - p_1)$$
$$\sin\sigma \cos(p_1' - \pi') = \sin s_1 \cos s_2 - \cos s_1 \sin s_2 \cos(p_2 - p_1).$$

These are exact formulas, but as s_1 and s_2 are always less than $12'$ in any real case that we have, we may generally put

$$p_1' = p_1, \text{ and } \pi' = \pi;$$
$$\cos s_1 = \cos s_2 = 1, \text{ and } \sin s_1 = s_1, \ \sin s_2 = s_2, \ \sin\sigma = \sigma,$$

and hence we get

$$\sigma \sin(p_1 - \pi) = s_2 \sin(p_2 - p_1)$$
$$\sigma \cos(p_1 - \pi) = s_1 - s_2 \cos(p_2 - p_1),$$

and from these we have

$$\sigma \sin \pi = s_1 \sin p_1 - s_2 \sin p_2$$
$$\sigma \cos \pi = s_1 \cos p_1 - s_2 \cos p_2.$$

Now $s_1 \sin p_1$ and $s_1 \cos p_1$ are the rectangular co-ordinates of the first satellite with respect to the center of the planet, and the preceding equations become

$$x = x_1 - x_2$$
$$y = y_1 - y_2,$$

where x and y are the relative co-ordinates of the satellites; a result that might have been anticipated, since the above assumptions reduce the work to that of plane co-ordinates. Should it be necessary in any case we may use the rigorous formulas; or, what is easier, we may expand them and retain terms of the lower orders. In the present case we have therefore

$$dx = dx_1 - dx_2$$
$$dy = dy_1 - dy_2.$$

The differentials on the right hand side can be expressed in terms of the differtials of the elements of the orbits of the two satellites, and of the differential coefficients with respect to their elements; and since the left hand side will be known by comparing the computed relative position with the observed, we have the means of forming the equations of condition for correcting the elements of these orbits, and therefore for determining the mass of the planet. Thus dx_1 and dx_2 will be of the form

$$dx_1 = \frac{dx_1}{da_1} \cdot \delta a_1 + \frac{dx_1}{d\varepsilon_1} \cdot \delta \varepsilon_1 + \frac{dx_1}{dP_1} \cdot \delta P_1 + \frac{dx_1}{de_1} \cdot \delta e_1 + \frac{dx_1}{dN_1} \cdot \delta N_1 + \frac{dx_1}{dJ_1} \cdot \delta J_1$$
$$dx_2 = \frac{dx_2}{da_2} \cdot \delta a_2 + \frac{dx_2}{d\varepsilon_2} \cdot \delta \varepsilon_2 + \frac{dx_2}{dP_2} \cdot \delta P_2 + \frac{dx_2}{de_2} \cdot \delta e_2 + \frac{dx_2}{dN_2} \cdot \delta N_2 + \frac{dx_2}{dJ_2} \cdot \delta J_2$$

where a_1, ε_1, P_1, e_1, N_1, J_1 are the elements of the orbit of the first satellite, and a_2, ε_2, P_2, e_2, N_2, J_2 are those of the second satellite. The equations of condition therefore that result from this method will each contain twelve unknown quantities; and it will be necessary to arrange the observations so that there may be no indetermination in the solution. For this purpose the satellites should be observed when they are on different sides of the planet as well as when they are on the same side, and in a variety of positions.

In the common method of referring a satellite directly to the center of the planet we have in the equations of condition six unknown quantities. Now the labor of solving a set of equations of condition increases rapidly with the number of unknown quantities. If i be this number, then the number of coefficients in the normal equations is found by the summation of an arithmetical series to be

$$\frac{i\,(i+3)}{2}.$$

The number of auxiliary quantities that have to be formed in the successive elimination of the unknown quantities from the normals is given in the following table, with their first and second differences.

No.	Δ_1	Δ_2
$\frac{i^2+i-2}{2}$,		
	$-i+0$	
$\frac{i^2-i-2}{2}$,		$+1$
	$-i+1$	
$\frac{i^2-3i+0}{2}$,		$+1$
	$-i+2$	
$\frac{i^2-5i+4}{2}$,		$+1$
	$-i+3$	
$\frac{i^2-7i+10}{2}$,		$+1$
	$-i+4$	
$\frac{i^2-9i+18}{2}$,		
etc.		

The law of this series is evident. The sum of $i-1$ terms of the series is

$$\text{Sum} = \frac{i(i-1)(i+2)}{2} - \frac{i^2(i-1)}{2} + \frac{i(i-1)(i-2)}{2.3}$$

$$= \frac{(i-1)\,i\,(i+4)}{2.3}$$

Hence the whole number of quantities to be formed in this solution is

$$\frac{(i-1)\,i\,(i+4)}{2.3} + \frac{i(i+3)}{2} = \frac{i(i+1)(i+5)}{2.3},$$

In the case of $i=6$, we shall have 77 quantities to form, and when $i=12$, we shall have 442 quantities to form. The labor of solution is nearly six times as great in the new method, but this is a consideration that is not perhaps entitled to much weight. Again, when the number of unknown quantities is increased, the probable error of a single equation is generally increased, but this increase will be small when the number of observations is large; and if the observations be well arranged, this question may be left out of consideration. On the other hand, the gain in accuracy seems to be substantial, and the new method promises a real advance, especially in the case of the satellites of Jupiter and Saturn. Another advantage is that it is a different method of making the observations, and in this way it offers an independent check on the old methods, and may be the means of eliminating some constant errors.